ε-ISOMETRY, ISOMETRY AND LINEAR ISOMETRY

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Abstract. Let X, Y be two real Banach spaces, and $\varepsilon \geq 0$. A map $f : X \to Y$ is said to be a standard ε-isometry if $|\|f(x)-f(y)\|-\|x-y\|| \leq \varepsilon$ for all $x, y \in X$ and with $f(0) = 0$. We say that a pair of Banach spaces (X, Y) is stable if there exists $\gamma > 0$ such that for every $\varepsilon > 0$ and every standard ε-isometry $f : X \to Y$ there is a bounded linear operator $T : L(f) \equiv \overline{\text{span}}\, f(X) \to X$ such that $\|Tf(x) - x\| \leq \gamma\varepsilon$ for all $x \in X$. $X(Y)$ is said to be universally left (right)-stable, if (X, Y) is always stable for every $Y(X)$. In this paper, we show first that if such an ε-isometry f exists, then there is a linear isometry $U : X^{**} \to Y^{**}$. Then we prove that universally right-stable spaces are just Hilbert spaces; every injective space is universally left-stable; Finally, we verify that a Banach space X which is linear isomorphic to a subspace of ℓ_∞ is universally left-stable if and only if it is linearly isomorphic to ℓ_∞; and a separable space X satisfying that (X, Y) is stable for every separable Y if and only if X is linearly isomorphic to c_0.

1. Introduction

In this paper, we have two goals: studying relationship between ε-isometry and linear isometry, and considering stability of ε-isometry of (real) Banach spaces. To begin with, we recall definitions of isometry and ε-isometry.

Definition 1.1. Let X, Y be two Banach spaces, $\varepsilon \geq 0$, and $f : X \to Y$ be a mapping.

(1) f is said to be an ε-isometry if

$$|\|f(x) - f(y)\| - \|x - y\|| \leq \varepsilon \ \text{ for all } x, y \in X;$$

(2) In particular, if $\varepsilon = 0$, then the 0-isometry f is simply called an isometry;

(3) We say that an (ε-) isometry f is standard if $f(0) = 0$.

Isometry and linear isometry. The study of properties of isometry between Banach spaces and its generalizations have continued for 80 years. The first celebrated result was due to Mazur and Ulam ([17],1932): Every surjective isometry between two Banach spaces is necessarily affine. But

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the simple example: $f : \mathbb{R} \to \ell_\infty^2$ defined for t by $f(t) = (t, \sin t)$ shows that it is not true if an isometry is not surjective. For non-surjective isometry, Figiel [8] showed the following remarkable theorem in 1968.

Theorem 1.2 (Figiel)**.** *Suppose f is a standard isometry from a Banach X to another Bnanach space Y. Then there is a linear operator $T : L(f) \to X$ with $\|T\| = 1$ such that $Tf(x) = x$, for all $x \in X$; or equivalently, $Tf = I_X$, the identity on X.*

In 2003, Godefroy and Kalton [10] studied the relationship between isometry and linear isometry, and showed the following deep theorem:

Theorem 1.3 (Godefroy-Kalton)**.** *Suppose that X, Y are two Banach spaces.*

(1) If X is separable and there is an isometry $f : X \to Y$, then Y contains an isometric linear copy of X;

(2) If X is a nonseparable weakly compactly generated space, then there exist a Banach space Y and an isometry $f : X \to Y$, so that X is not linearly isomorphic any subspace of Y.

ε-isometry and stability. In 1945, Hyers and Ulam proposed the following question[13] (see, also [19]): whether for every surjective ε-isometry $f : X \to Y$ with $f(0) = 0$ there exist a surjective linear isometry $U : X \to Y$ and $\gamma > 0$ such that

$$\|f(x) - Ux\| \leq \gamma\varepsilon, \text{ for all } x \in X. \tag{1.1}$$

After many years efforts of a number of mathematicians (see, for instance, [9], [11], [13], and [19]), the following sharp estimate was finally obtained by Omladič and Šemrl [19].

Theorem 1.4 (Omladič-Šemrl)**.** *If $f : X \to Y$ is a surjective ε-isometry with $f(0) = 0$, then there is a surjective linear isometry $U : X \to Y$ such that*

$$\|f(x) - Ux\| \leq 2\varepsilon, \text{ for all } x \in X.$$

The study of nonsurjective ε-isometry has also brought to mathematicians' attention (see, for instance [5], [19], [20], [21] and [23]). Qian[20] first proposed the following problem in 1995.

Problem 1.5. *Whether there exists a constant $\gamma > 0$ depending only on X and Y with the following property: For each standard ε-isometry $f : X \to Y$ there is a bounded linear operator $T : L(f) \to X$ such that*

$$\|Tf(x) - x\| \leq \gamma\varepsilon, \quad \text{for all } x \in X. \tag{1.2}$$

Then he showed that the answer is affirmative if both X and Y are L_p spaces. Šemrl and Väisälä [21] further presented a sharp estimate of (1.2) with $\gamma = 2$ if both X and Y are L^p spaces for $1 < p < \infty$.

However, Qian (in the same paper [20]) presented a simple counterexample showing that if a separable Banach space Y contains a uncomplemented closed subspace X then for every $\varepsilon > 0$ there is a standard ε-isometry

$f : X \to Y$ which is unstable. This disappointment makes us to search for (1) some weaker stability version and (2) some appropriate complementability assumption on some subspaces of Y associated with the mapping. Cheng, Dong and Zhang [3] showed the following theorems about the two questions.

Theorem 1.6 (Cheng-Dong-Zhang)**.** *Let X and Y be Banach spaces, and let $f : X \to Y$ be a standard ε-isometry for some $\varepsilon \geq 0$. Then for every $x^* \in X^*$, there exists $\phi \in Y^*$ with $\|\phi\| = \|x^*\| \equiv r$ such that*

$$|\langle \phi, f(x)\rangle - \langle x^*, x\rangle| \leq 4\varepsilon r, \text{ for all } x \in X. \tag{1.3}$$

For a standard ε-isometry $f : X \to Y$, let

$Y \supset E =$ the annihilator of all bounded linear functionals and bounded on

$$C(f) \equiv \overline{\text{co}}(f(X), -f(X)),$$

i.e.

$$E = \{y \in Y : \langle y^*, y\rangle = 0, y^* \in Y^* \text{ is bounded on } C(f)\}. \tag{1.4}$$

Theorem 1.7 (Cheng-Dong-Zhang)**.** *Let X and Y be Banach spaces, and let $f : X \to Y$ be a standard ε-isometry for some $\varepsilon \geq 0$. Then*

(i) If Y is reflexive and if E is α-complemented in Y, then there is a bounded linear operator $T : Y \to X$ with $\|T\| \leq \alpha$ such that

$$\|Tf(x) - x\| \leq 4\varepsilon, \quad \textit{for all } x \in X.$$

(ii) If Y is reflexive, smooth and locally uniformly convex, and if E is α-complemented in Y, then there is a bounded linear operator $T : Y \to X$ with $\|T\| \leq \alpha$ such that the following sharp estimate holds

$$\|Tf(x) - x\| \leq 2\varepsilon, \quad \textit{for all } x \in X.$$

Universal stability spaces of ε-isometries. For study of the stability of ε-isometry of Banach spaces, the following two questions are very natural.

Problem 1.8. *Is there a characterization for the class of Banach spaces X satisfying that for every Banach space Y there exists a positive number γ such that the following conclusion holds: for every standard ε-isometry $f : X \to Y$, there exists a bounded linear operator $T : L(f) \to X$ such that*

$$\|Tf(x) - x\| \leq \gamma\varepsilon, \quad \textit{for all } x \in X,$$

that is, inequality (1.2) holds. Every space X in this class is said to be a universal left-stability space.

Problem 1.9. *Can we characterize the class of Banach spaces Y, such that for every Banach space X there exists a positive number γ such that the following conclusion holds: for every standard ε-isometry $f : X \to Y$ there exists a bounded linear operator $T : L(f) \to X$ such that*

$$\|Tf(x) - x\| \leq \gamma\varepsilon, \quad \textit{for all } x \in X.$$

Every space Y in this class is called a universal right-stability space.

In this paper, we first consider the relationship between ε-isometry and linear isometry. Using Cheng, Dong and Zhang's lemma (Theorem 1.6) and existence of invariant means of Abelian semigroups, we show that for every standard ε-isometry $f : X \to Y$, there is a w^*-to-w^* continuous linear isometry $U : X^{**} \to Y^{**}$. In particular, if Y is reflexive, then Y contains a linear isometric copy of X. This is a complement and partial extension of the Godefroy-Kalton theorem (Theorem 1.3). Then we study universal left-stability and universal right-stability of Banach spaces. As a result, with the help of Qian's counterexample and Theorem 1.6, incorporating of the famous result of Lindenstrauss and Tzafriri [16]: A Banach space satisfying that every closed subspace is complemented is isomorphic to a Hilbert space, we show, up to linear isomorphism, right stability spaces are just Hilbert spaces. (See also [3], but without proof.) Using Goodner-Kelley-Nachbin's theorem ([12], [18] and [14]) and Zippin's theorem [26] we prove that every injective space is universally left-stable, and a Banach space X which is linear isomorphic to a subspace of ℓ_∞ is universally left-stable if and only if it is linearly isomorphic to ℓ_∞. Finally, we verify that separable space X satisfying that (X, Y) is stable for every separable Y if and only if X is linearly isomorphic to c_0.

All symbols and notations in this paper are standard. We use X to denote a real Banach space and X^* its dual. B_X and S_X denote the closed unit ball and the unit sphere of X, respectively. For a subspace $E \subset X$, $E^\perp$ denotes the annihilator of E, i.e. $E^\perp = \{x^* \in X^* : \langle x^*, e\rangle = 0 \text{ for all } e \in E\}$. Given a bounded linear operator $T : X \to Y$, $T^* : Y^* \to X^*$ stands for its conjugate operator. For a subset $A \subset X$ (X^*), $\overline{A}$, (w^*-$\overline{A}$) and $\mathrm{co}(A)$ stand for the closure (the w^*-closure), and the convex hull of A, respectively.

2. ε-isometry and linear isometry

Suppose that X, Y are Banach spaces, and $f : X \to Y$ is a standard ε-isometry for some $\varepsilon \geq 0$. Let $\Delta : X^* \to 2^{Y^*}$ be defined for $x^* \in X^*$ by

$$\Delta(x^*) = \{\phi \in Y^* : \|\phi\| = \|x^*\|, |\langle x^*, x\rangle - \langle\phi, f(x)\rangle| \leq 4\|x^*\|\varepsilon, \quad \forall x \in X\}. \tag{2.1}$$

By Theorem 1.6, $\Delta(x^*)$ is nonempty for each $x^* \in X^*$.

Recall

$$C(f) = \overline{\mathrm{co}}\{f(X) \cup -f(X)\}, \tag{2.2}$$

$$M_\varepsilon = \{\phi \in Y^* : \exists \beta > 0 \text{ such that } \phi(y) \leq \beta\varepsilon \ \forall\, y \in C\}, \tag{2.3}$$

and let

$$M = \overline{M_\varepsilon}, \quad \text{the closure of } M_\varepsilon. \tag{2.4}$$

Lemma 2.1 (Cheng-Dong-Zhang [3], Theorem 4.4)**.** *With the spaces X, Y and with the standard ε-isometry f as above, we define $Q : X^* \to Y^*/M$ by*

$$Q(x^*) = \Delta(x^*) + M, \quad x^* \in X^*. \tag{2.5}$$

Then Q is a linear isometry.

In this section, we first show the following theorem.

Theorem 2.2. *With the ε-isometry f, the associated subspace $M \subset Y^*$ and the linear isometry $Q : X^* \to Y^*/M$ as above, then*

(i) there is a linear operator $R : Y^*/M \to X^*$ such that $\|R\| \leq 1$ and $RQ = I_{X^*}$;

(ii) the conjugate operator R^* of R is a linear isometry from X^{**} to $M^\perp \subset Y^{**}$.

We will use Theorem 1.6, Lemma 2.1, and invariant mean procedure to show the theorem. Before starting the proof we recall definition of (left) mean of a semigroup and some related result, which are taken from Benyamini and Lindenstrauss' book [2] (pp.417-418).

Definition 2.3. Let G be a semigroup. A left-invariant mean on G is a linear functional μ on $\ell_\infty(G)$ such that:

(i) $\mu(1) = 1$,

(ii) $\mu(f) \geq 0$ for every $f \geq 0$,

(iii) $\forall f \in l_\infty(G), \forall g \in G, \mu(f_g) = \mu(f)$, where f_g is the left-translation of f by g; i.e., $f_g(h) = f(gh), \forall h \in G$.

(iv) Analogously, we can define right-invariant mean of G. An invariant mean is a linear functional on $\ell_\infty(G)$ which is both left-invariant and right-invariant.

Clearly, an invariant mean of a semigroup G is just an index-translation invariant positive functional of norm one on $\ell_\infty(G)$.

Note that (i) and (ii) are equivalent to (i) and $\|\mu\| = 1$.

Lemma 2.4. *Every Abelian semigroup G (in particular, every linear space) has an invariant mean.*

Proof of Theorem 2.2.

Let X, Y be Banach spaces and $f : X \to Y$ be a standard ε-isometry.

(i). Note X is an Abelian group with respect to the vector addition of X. By Lemma 2.4, there exists a translation invariant mean μ on $\ell_\infty(X)$. Fix any $x \in X$. Since f is an ε-isometry,

$$g_x(z) = f(x+z) - f(z), \quad \text{for all } z \in X \tag{2.6}$$

defines a bounded mapping $g_x : X \to Y$. Therefore, $\langle \phi, g_x \rangle \in \ell_\infty(X)$ for every $\phi \in Y^*$. We also denote the invariant mean by μ_z or $\mu_z(\cdot)$, emphasizing that the mean is taken with respect to the variable z.

Next, we define a linear mapping $R : Y^* \to \mathbb{R}^X$ by

$$\langle R\phi, x \rangle = \mu(\langle \phi, g_x \rangle), \quad \text{for all } \phi \in Y^*, x \in X. \tag{2.7}$$

We claim that

(1) $R\phi \in X^*$ for every $\phi \in Y^*$;

(2) $\|R\phi\| \leq \|\phi\|$ for every $\phi \in Y^*$;

(3) $M \subset \ker R$.

Given $u, v \in X$,

$$\begin{aligned}(2.8)\qquad \langle R\phi, u+v\rangle &= M(\langle \phi, g_{u+v}\rangle) = \mu_z(\langle \phi, f(u+v+z)-f(z)\rangle)\\ &= \mu_z(\langle \phi, f(u+v+z) - f(v+z)\rangle) + \mu_z(\langle \phi, f(v+z) - f(z)\rangle)\\ &= \mu_z(\langle \phi, f(u+z) - f(z)\rangle) + \mu_z(\langle \phi, f(v+z) - f(z)\rangle)\\ &= \mu(\langle \phi, g_u\rangle) + \mu(\langle \phi, g_v\rangle) = \langle R\phi, u\rangle + \langle R\phi, v\rangle.\end{aligned}$$

That is, additivity of $R\phi$ has been shown. It follows from additivity of $R\phi$,

$$(2.9)\qquad \langle R\phi, \lambda u\rangle = \lambda\langle R\phi, u\rangle, \quad \text{for all rational number } \lambda.$$

Therefore, (2.6), (2.7) and (2.9) imply for all $u, v \in X$ and $k \in \mathbb{N}$,

$$\begin{aligned}&|\langle R\phi, u\rangle - \langle R\phi, v\rangle| = \frac{1}{k}|\langle R\phi, ku\rangle - \langle R\phi, kv\rangle|\\ &= |\mu_z(\langle \phi, \frac{(f(ku+z) - f(z))}{k}\rangle) - \mu_z(\langle \phi, \frac{(f(kv+z) - f(z))}{k}\rangle)|\\ &= |\mu_z(\langle \phi, \frac{(f(ku+z) - f(kv+z))}{k}\rangle)| \le \|\mu\|\|\phi\|\|\frac{(f(ku+z) - f(kv+z))}{k}\|\\ &\le \|\mu\|\|\phi\|\frac{\|(ku+z) - (kv+z)\| + \varepsilon}{k} = \|\phi\|\|u-v\| + \frac{\varepsilon}{k}\\ &\to \|\phi\|\|u-v\|, \text{ as } k \to \infty.\end{aligned}$$

Hence

$$(2.10)\qquad |\langle R\phi, u\rangle - \langle R\phi, v\rangle| \le \|\phi\|\|u-v\|, \text{ for all } u, v \in X.$$

We have proven that $R\phi$ is continuous on X. (2.8),(2.9) and (2.10) together imply that $R\phi$ is linear and with $\|R\phi\| \le \|\phi\|$, that is, (1) and (2) hold. And this further entails that $R : Y^* \to X^*$ is a linear operator with $\|R\| \le 1$.

To show (3), continuity of R allows us only to verify $M_\varepsilon \subset \ker R$. Given $\phi \in M_\varepsilon$, definition of M_ε (i.e. (2.3)) says that $\langle \phi, f\rangle \in \ell_\infty(X)$. For every $x \in X$, it follows from the translation invariance of μ,

$$\begin{aligned}\langle R\phi, x\rangle &= \mu_z(\langle \phi, f(x+z) - f(z)\rangle)\\ &= \mu_z(\langle \phi, f(x+z)\rangle) - \mu_z(\langle \phi, f(z)\rangle) = 0.\end{aligned}$$

Consequently, (3) holds. We have shown that R is eventually from Y^*/M to X^* with $\|R\| \le 1$.

It remains to show $RQ = I_{X^*}$. Let $x^* \in X^*$. Note $Qx^* = \Delta x^* + M$.

$$RQx^* = R(\Delta x^* + M) = R(\Delta x^*)$$

By (2.1), for each $\phi \in \Delta(x^*)$, we have

$$|\langle \phi, f(z)\rangle - \langle x^*, z\rangle| \le 4\varepsilon\|x^*\|, \text{ for all } z \in X.$$

Therefore, for all $x \in X$,

$$\begin{aligned}&\langle R\phi, x\rangle = \mu_z(\langle \phi, f(x+z)-f(z)\rangle)\\ &= \mu_z\{(\langle \phi, f(x+z)\rangle - \langle x^*, x+z\rangle) - (\langle \phi, f(z)\rangle - \langle x^*, z\rangle) + \langle x^*, x\rangle\}\\ &\le \mu(4\varepsilon\|x^*\|)+ \le \mu(4\varepsilon\|x^*\|) + \mu_z(\langle x^*, x\rangle)\\ &= 8\varepsilon\|x^*\| + \langle x^*, x\rangle\end{aligned}$$

or, equivalently,

$$\langle R\phi - x^*, x\rangle \le 8\varepsilon\|x^*\| \ \text{ for all } x \in X.$$

Consequently, $R\phi - x^* = 0$ for all $\phi \in \Delta(x^*)$. Therefore,

$$RQ(x^*) = R\Delta(x^*) = x^*. \tag{2.11}$$

(ii). By the facts $\|R\|, \|Q\| \le 1$, and $RQ = I_{X^*}$ that we have just proven, we obtain $Q^* : (Y^*/M)^* = M^\perp \to X^{**}$, $R^* : X^{**} \to M^\perp$ with $\|R^*\| = \|R\| \le 1$, $\|Q^*\| = \|Q\| \le 1$ and $Q^*R^* = (RQ)^* = I_{X^{**}}$. Therefore, for all $x^{**} \in X^{**}$, we have

$$\|x^{**}\| = \|(Q^*R^*)x^{**}\| \le \|Q^*\|\|R^*(x^{**})\| \le \|R^*(x^{**})\|.$$

On the other hand,

$$\|R^*(x^{**})\| \le \|R^*\|\|x^{**}\| \le \|x^{**}\|.$$

Hence, $R^* : X^{**} \to M^\perp$ is a linear isometry.

Corollary 2.5. *Let f be an ε-isometry from a Banach space X to another Banach space Y. If Y is reflexive, then X must be linearly isometric to a subspace of Y.*

Remark 2.6. By Godefroy-Kalton's theorem (Theorem 1.3), we can not claim that the linear isometry $R^* : X^{**} \to Y^{**}$ in Theorem 2.2 satisfying $R^*|_X$ is also a linear isometry from X to Y in general, until R is a w^*-to-w^* continuous, or equivalently, R is a conjugate operator. Corollary 2.5 is a complement of Godefroy-Kalton's theorem in the case that X is not separable, but we do not know whether Y contains a isometric copy of X if X is separable.

3. Universal stability spaces for ε-isometries

In this section, we search for some properties of the class of universal left (right)-stability spaces for ε-isometries.

Recall that a Banach space $X(Y)$ is universally left(right)-stable if it satisfies that for every Banach space Y (X) and every standard ε-isometry $f : X \to Y$, there exist a bounded linear operator $T : L(f) \to X$ and a positive number γ such that

$$\|Tf(x) - x\| \le \gamma\varepsilon, \quad for\ all\ x \in X. \tag{3.1}$$

As a result we show (1) inequality (3.1) holds for every Banach space X if and only if Y is, up to linear isomorphism, a Hilbert space; (2) An injective Banach space is always left-stable; (3) A Banach space linear isomorphic to a subspace of ℓ_∞ is universally left-stable if and only if it is linearly isomorphic to ℓ_∞ and (4) for a separable Banach space X, inequality (3.1) holds for every separable Banach space Y if and only if X is a separably injective Banach space.

The following lemma was motivated by Qian's counterexample.

Lemma 3.1. *Let X be a closed subspace of a Banach space Y. If card(X) = card(Y), then for every $\varepsilon > 0$ there is a standard ε-isometry $f : X \to Y$ such that*

(1) $L(f) \equiv \overline{span}\, f(X) = Y$;

(2) X is complemented whenever f is stable.

Proof. Without loss of generality we assume $X \neq \{0\}$.

(1) Since card(X) = card(Y), there exists a bijection $g : X \to B_Y$ with $g(0) = 0$. Let $f : X \to Y$ be defined by

$$f(x) = x + \frac{\varepsilon}{2} g(x), \text{ for all } x \in X.$$

Then f is a standard ε-isometry. To show $L(f) \equiv \overline{\text{span}}\, f(X) = Y$, it suffices to prove $X \subset L(f)$. Given $x \in X$, let $x_n = f(nx)/n = x + \frac{\varepsilon}{2n} g(nx)$. Then $\|x_n - x\| \leq \frac{\varepsilon}{2n} \to 0$. Hence, $x \in L(f)$.

(2) Let $T : Y \to X$ be a bounded linear operator and $\gamma > 0$ be a constant such that

$$\|Tf(x) - x\| \leq \gamma\varepsilon, \quad \text{for all } x \in X.$$

Then it is easy to observe that T is a projection, hence, X is complemented in Y. □

Theorem 3.2. *A Banach space Y is universally right-stable if and only if it is isomorphic to a Hilbert space.*

Proof. Sufficiency. Suppose that Y is linearly isomorphic to a Hilbert space and X is a Banach space. Given $\varepsilon \geq 0$ and a standard ε-isometry $f : X \to Y$. Since Y is reflexive and since every closed subspace of Y is complemented, by Theorem 4.8 of [3], inequality (3.1) holds, i.e., Y is universal right-stable.

Necessity. By definition of universal right-stability, every closed subspace of Y is again universal right-stable. Fix any closed separable subspace Z of Y. By Lemma 3.1, universal right-stability of Z entails that every closed subspace of Z is complemented in Z. According to Lindenstrauss-Tzafriri's theorem, Z is isomorphic to a (separable) Hilbert space. Hence, Y itself is isomorphic to a Hilbert space.

□

A Banach space X is said to be injective if it has the following extension property: Every bounded linear operator from a closed subspace of a Banach space into X can be extended to be a bounded operator on the whole space (see, for instance, [1]). X is called isometrically injective if every such bounded operator has a norm-preserved extension. Goondner [12] introduced a family of Banach spaces coinciding with the family of injective spaces: for any $\lambda \geq 1$, a Banach space X is a P_λ-space if, whenever X is isometrically embedded in another Banach space, there is a projection onto the image of X with norm not larger than λ. The following result was due to Day [4] (see, also, Wolfe [25], Fabian et al. [7], p. 242).

Proposition 3.3. *A Banach space X is (isometrically) injective if and only if it is a P_λ (P_1)-space for some $\lambda \geq 1$.*

Goondner [12], Nachbin [18] and Kelley [14] characterized the isometrically injective spaces.

Theorem 3.4 (Goodner-Kelley-Nachbin, 1949-1952)**.** *A Banach space is isometrically injective if and only if it is isometrically isomorphic to the space of continuous functions $C(K)$ on an extremely disconnected compact Hausdorff space K, i.e. the space K such that the closure of any open set is open in K.*

Remark 3.5. For any set Γ, that $\ell_\infty(\Gamma)$ is an isometrically injective space follows from the Hahn-Banach theorem applied coordinatewise.

Theorem 3.6. *Let X be an injective Banach space. Then for every Banach space Y, there exists a positive number γ such that for nonnegative number ε and for every standard ε-isometry $f : X \to Y$ there is a bounded linear operator $T : Y \to X$ satisfying*

$$\|Tf(x) - x\| \le \gamma\varepsilon, \quad \text{for all } x \in X.$$

Proof. Assume that X is an injective Banach space. We can assume that X is a closed complemented subspace of $\ell_\infty(\Gamma)$; otherwise, we can identify X for $J_X(X)$ as a complemented subspace of $\ell_\infty(\Gamma)$, where Γ denotes the closed ball B_{X^*} of X^*. Let $P : \ell_\infty(\Gamma) \to X$ be a projection such that $\|P\| \equiv \alpha < \infty$. Given any $\beta \in \Gamma$, let $\delta_\beta \in \ell_\infty(\Gamma)^*$ be defined for $x = (x(\gamma))_{\gamma\in\Gamma} \in \ell_\infty(\Gamma)$ by $\delta_\beta(x) = x(\beta)$. Assume that $f : X \to Y$ be an ε-isometry with $f(0) = 0$. For every $x^* \in X^*$, by Theorem 1.6, there is $\phi \in Y^*$ with $\|\phi\| = \|x^*\|$ such that

$$|\langle\phi, f(x)\rangle - \langle x^*, x\rangle| \le 4\varepsilon\|x^*\|, \text{ for all } x \in X. \tag{3.2}$$

In particular, letting $x^* = \delta_\gamma$ in (3.2) for every fixed $\gamma \in \Gamma$, we obtain a linear functional $\phi_\gamma \in Y^*$ satisfying (3.2) with $\|\phi_\gamma\| = \|\delta_\gamma\|_X \le 1$. Therefore, $(\phi_\gamma(y))_{\gamma\in\Gamma} \in \ell_\infty(\Gamma)$ for every $y \in Y$.

Let $T(y) = P(\phi_\gamma(y))_{\gamma\in\Gamma}$, for all $y \in Y$, and note $P|_X = I_X$, the identity from X to itself. Then $\|T\| \le \|P\| = \alpha$ and for all $x \in X$,

$$\|Tf(x) - x\| = \|P(\phi_\gamma(f(x)))_{\gamma\in\Gamma} - (\delta_\gamma(x))_{\gamma\in\Gamma}\|$$

$$= \|P(\phi_\gamma(f(x)))_{\gamma\in\Gamma} - P((\delta_\gamma(x))_{\gamma\in\Gamma})\|$$

$$\le \|P\| \cdot \|(\phi_\gamma(f(x)))_{\gamma\in\Gamma} - (\delta_\gamma(x))_{\gamma\in\Gamma}\|_\infty \le 4\alpha\varepsilon.$$

We finish the proof by taking $\gamma = 4\alpha$.

□

Corollary 3.7. *Every finite dimensional normed space is universally-stable.*

Proof. Since every finite dimensional normed space is linearly isomorphic to an Euclidean space, Theorem 3.2 entails that it is universally right-stable. While Theorem 3.6 implies that it is also universally left-stable because that every finite dimensional normed space is necessarily injective. □

Theorem 3.8. *An infinite dimensional Banach space X which is linear isomorphic to a subspace of ℓ_∞ is universally left-stable if and only if it is linearly isomorphic to ℓ_∞.*

Proof. Sufficiency. Suppose that X is linearly isomorphic to ℓ_∞. Since ℓ_∞ is injective (Remark 3.5), X is necessarily injective. Therefore, this is a direct consequence of Theorem 3.6.

Necessity. Since $\dim X = \infty$ and since it is isomorphic to a subspace of ℓ_∞, we have

$$card(X) \geq \aleph = \aleph^{\mathbb{N}} = card(\mathbb{R}^{\mathbb{N}}) = card(\ell_\infty) \geq card(X). \tag{3.3}$$

Assume that X is universally left-stable. We claim that X is linearly isomorphic to ℓ_∞. Since ℓ_∞ is, up to linear isomorphism, the minimum (infinite dimensional) injective space contained in ℓ_∞ and since X is linearly isomorphic to a subspace of ℓ_∞, by Lindenstrauss' theorem [15] (i.e. ℓ_∞ is a prime space), it suffices to show that X is injective.

Suppose, to the contrary, that X is not injective. Let $T : X \to \ell_\infty$ be a linear embedding. Then $Z \equiv T(X)$ is not a complemented subspace of ℓ_∞. We can put an equivalent norm $|||\cdot|||$ on ℓ_∞ such that X is isometric to $(Z, |||\cdot|||)$. Indeed, let $|\cdot|$ on Z be defined by $|z| = \|x\|$ for all $z = Tx \in Z$. Then, we choose a sufficiently large $\lambda > 0$ and define $|||\cdot|||$ on ℓ_∞ by $|||u||| = \inf\{|v| + \lambda\|u - v\| : v \in Z\}$. Clearly, the norm $|||\cdot|||$ has the property we desired. By Lemma 3.1, there is a standard ε-isometry $f : X \to (\ell_\infty, |||\cdot|||)$ which is not stable. □

Remark 3.9. We do not know whether the theorem above still holds in general, i.e., whether the universal left-stability of X entails that X is injective.

Remark 3.10. Theorem 3.2 says that universal right-stability is isomorphic invariant. It follows from Theorem 3.8 that the universal left-stability of a Banach space isomorphic to a subspace of ℓ_∞ is also isomorphically invariant. But we do not know whether it is isomorphically invariant in general.

A separable Banach space X is said to be separably injective if it has the following extension property: Every bounded linear operator from a closed subspace of a separable Banach space into X can be extended to be a bounded operator on the whole space. In 1941 Sobczyk [22] showed that c_0 is separably injective, and in 1977, Zippin [26] proved that c_0 is, up to isomorphism, the only separable space which is separably injective.

With the aim of Zippin's theorem, the following theorem says that c_0 is (up to isomorphism) the only space satisfying inequality (3.1) for every separable Y.

Theorem 3.11. *Let X be a separable Banach space. Then it satisfies that for every separable Banach space Y there is $\gamma > 0$ such that for every nonnegative number ε and for every standard ε-isometry $f : X \to Y$ there exists a bounded linear operator $T : L(f) \to X$ such that*

$$\|Sf(x) - x\| \leq \gamma\varepsilon, \quad \text{for all } x \in X$$

if and only if X is linearly isomorphic to c_0, or equivalently, X is a separably injective space.

Proof. Sufficiency. Let X be a Banach space isomorphic to c_0 and $T : X \to c_0$ be a linear isomorphism. Assume that $(e_n)_{n=1}^{\infty}$ is the canonical basis of c_0 with the standard biorthogonal functionals $(e_n^*)_{n=1}^{\infty} \subset \ell_1$. Let $(x_n) \subset X$ satisfy $Tx_n = e_n$ for all $n \in \mathbb{N}$, and let $T^* : \ell_1 \to X^*$ be the conjugate operator of T. Then

$$Tx = \sum (T^* e_n^*)(x) e_n \text{ and } x = \sum (T^* e_n^*)(x) T^{-1} e_n, \quad \text{for all } x \in X.$$

Let $\alpha = \|T\| \cdot \|T^{-1}\|$, $x_n^* = T^* e_n^* \in \|T\| B_{X^*}$ for all $n \in \mathbb{N}$, and note $x_n = T^{-1} e_n \in X$. By Theorem 1.7 there exists $\phi_n \in \|T\| B_{Y^*}$ with $\|\phi_n\| = \|x_n^*\|$ such that

$$|\langle \phi_n, f(x)\rangle - \langle x_n^*, x\rangle| \leq 4\varepsilon \|T\|, \text{ for all } x \in X. \tag{3.4}$$

Since $e_n^* \to 0$ in the w^*-topology of $\ell_1 = c_0^*$, $x_n^* = T^* e_n^* \to 0$ in the w^*-topology of X^*. Let

$$K = \{\psi \in \|T\| B(Y^*) : |\langle \psi, f(x)\rangle| \leq 4\varepsilon \|T\|, \text{ for all } x \in X\}. \tag{3.5}$$

Then K is a nonempty w^*-closed compact subset of Y^*. Since Y is separable, $(\|T\| B_{Y^*}, w^*)$ is metrizable. Let ρ be a metric such that $(\|T\| B_{Y^*}, \rho)$ is isomorphic to $(\|T\| B_{Y^*}, w^*)$. Since $(\|T\| B_{Y^*}, \rho)$ is a compact metric space and since K is a compact subset of it, $(\phi_n) \subset K$ has at least one ρ-sequentially cluster point. Since (x_n^*) is a w^*-null sequence in X^*, inequality (3.4) entails that any ρ-cluster point ϕ of (ϕ_n) is in K and with $\|\phi\| \leq \|T^*\| = \|T\|$. This further implies that $\operatorname{dist}_\rho(\phi_n, K) \to 0$. Consequently, there is a sequence $(\psi_n) \subset K$ such that $\operatorname{dist}_\rho(\phi_n, \psi_n) \to 0$, or equivalently, $\phi_n - \psi_n \to 0$ in the w^*-topology of Y^*. Hence, for every $y \in Y$,

$$Uy \equiv \sum_{n=1}^{\infty} \langle \phi_n - \phi, y\rangle e_n \in c_0 \tag{3.6}$$

and with

$$\|Uy\| \leq (\sup_{n \in \mathbb{N}} \|\phi_n - \psi_n\|)\|y\| \leq 2\|T\|\|y\|, \tag{3.7}$$

that is, $\|U\| \leq 2\|T\|$.

Finally, let

$$S(y) = T^{-1}(Uy) = \sum_{n=1}^{\infty} \langle \phi_n - \psi_n, y\rangle x_n \text{ for all } y \in Y. \tag{3.8}$$

Then

$$\|S\| = \|T^{-1} U\| \leq 2\|T\| \cdot \|T^{-1}\| = 2\alpha$$

and

$$\|Sf(x)-x\| = \|\sum_{n=1}^{\infty}\langle \phi_n-\psi_n, f(x)\rangle x_n - \sum_{n=1}^{\infty}\langle x_n^*, x\rangle x_n\|$$
$$= \lim_{n\to\infty}\|\sum_{i=1}^{n}\langle \phi_i-\psi_i, f(x)\rangle x_i - \sum_{i=1}^{n}\langle x_i^*, x\rangle x_i\|$$
$$= \lim_{n\to\infty}\|\sum_{i=1}^{n}(\langle \phi_i, f(x)\rangle - \langle x_i^*, x\rangle)x_i - \sum_{i=1}^{n}\langle \psi_i, f(x)\rangle x_i\|$$
$$\le \limsup_{n\to\infty}\|\sum_{i=1}^{n}(\langle \phi_i, f(x) - \langle x_i^*, x\rangle)x_i\| + \limsup_{n\to\infty}\|\sum_{i=1}^{n}\langle \psi_i, f(x)\rangle x_i\|$$
$$= \limsup_{n\to\infty}\|T^{-1}\sum_{i=1}^{n}(\langle \phi_i, f(x)\rangle - \langle x_i^*, x\rangle)e_i\| + \limsup_{n\to\infty}\|T^{-1}(\sum_{i=1}^{n}\langle \psi_i, f(x)\rangle e_i)\|$$
$$\le \|T^{-1}\|\cdot\limsup_{n\to\infty}(\|\sum_{i=1}^{n}(\langle \phi_i, f(x)\rangle - \langle x_i^*, x\rangle)e_i\| + \|\sum_{i=1}^{n}\langle \psi_i, f(x)\rangle e_i)\|)$$
$$\le \|T^{-1}\|(\sup_n|\langle \phi_i, f(x)\rangle - \langle x_i^*, x\rangle| + \sup_n|\langle \psi_i, f(x)\rangle|)$$
$$\le 8\varepsilon\|T\|\cdot\|T^{-1}\| = 8\varepsilon\alpha.$$

We finish the proof of sufficiency by taking $\gamma = 8\alpha$.

Necessity. If X is not linearly isomorphic to c_0, then by Zippin's theorem, X is not separably injective. Therefore, there exists a separable Banach space Y, which contains X as its a uncomplemented subspace. Clearly, $\text{card}(X) = \text{card}(Y)$. By Lemma 3.1 again, for every $\varepsilon > 0$, there is a standard ε-isometry $f : X \to Y$ which is not stable. □

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References

[1] F. Albiac, N.J. Kalton, Topics in Banach Space Theory, Graduate Texts in Mathematics 233, Springer, New York, 2006.

[2] Y. Benyamini, J. Lindenstrauss, Geometric Nonlinear Functional Analysis I, Amer. Math. Soc. Colloquium Publications, Vol.48, Amer. Math. Soc., Providence, RI, 2000.

[3] L. Cheng, Y. Dong, W. Zhang, On stability of Nonsurjective ε-isometries of Banach spaces, J. Funct. Anal. 264(2013), 713-734.

[4] M.M. Day, Normed Linear Spaces, Berlin, 1958.

[5] S.J. Dilworth, Approximate isometries on finite-dimensional normed spaces, Bull. London Math. Soc. 31(1999), 471-476.

[6] Y. Dutrieux, G. Lancien, Isometric embeddings of compact spaces into Banach spaces, J. Funct. Anal. 255 (2008), 494-501.

[7] M. Fabian, P. Habala, P. H¢jek, V. Montesinos, V., Zizler, Banach Space Theory, CMS Books in Mathematics, 1st Edition, 2011.

[8] T. Figiel, On non linear isometric embeddings of normed linear spaces, Bull. Acad. Polon. Sci. Math. Astro. Phys.16 (1968), 185-188.

[9] J. Gevirtz, Stability of isometries on Banach spaces, Proc. Amer. Math. Soc.89 (1983), 633-636.
[10] G. Godefroy, N.J. Kalton, Lipschitz-free Banach spaces, Studia Math. 159 (2003), 121-141.
[11] P.M. Gruber, Stability of isometries, Trans. Amer. Math. Soc. 245 (1978), 263-277.
[12] D.B. Goodner, Projections in normed linear spaces, Trans. Amer. Math. Soc. 69 (1950), 89-108.
[13] D.H. Hyers, S. M. Ulam, On approximate isometries, Bull. Amer. Math. Soc.51 (1945), 288-292.
[14] J.L. Kelley, Banach spaces with the extension property, Trans. Amer. Math. Soc. 72 (1952), 323-326.
[15] J. Lindenstrauss, On complemented subspaces of m, Israel J. Math. 5 (1967), 153-156.
[16] J. Lindenstrauss, L. Tzafriri, Classical Banach Spaces (I), Springer-Verlag, Berlin, Heidelberg, New York, 1977.
[17] S. Mazur, S. Ulam, Sur les transformations isométriques d'espaces vectoriels normés, C.R. Acad. Sci. Paris 194 (1932), 946-948.
[18] L. Nachbin, On the Han-Banach theorem, An. Acad. Bras. Cienc. 21 (1949), 151-154.
[19] M. Omladič, P. Šemrl, On non linear perturbations of isometries, Math. Ann. 303 (1995), 617-628.
[20] S. Qian, ε-Isometric embeddings, Proc. Amer. Math. Soc. 123 (1995) 1797-1803.
[21] P. Šemrl, J. Väisälä, Nonsurjective nearisometries of Banach spaces, J. Funct. Anal. 198 (2003), 268-278.
[22] A. Sobczyk, Projection of the space (m) on its subspace c_0, Bull. Amer. Math. Soc. 47 (1941), 938-947.
[23] J. Tabor, Stability of surjectivity, J. Approx. Theory 105 (2000) 166-175.
[24] R. Villa, Isometric embedding into spaces of continuous functions, Studia Math. 129 (1998), 197-205.
[25] J. Wolfe, Injective Banach spaces of type $C(T)$, Israel J. Math. 18 (1974), 133-140.
[26] M. Zippin, The separable extension problem, Israel J. Math. 26 (1977), 372-387.

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